Study Guide for Final Exam
Calculus II
In-Class Exam on [Exam Day, Date, and Time]

The Final Exam will be a cumulative test covering concepts and calculations that we have studied throughout this semester. You may use a calculator during this test; you must show me that you've cleared the memory before the exam begins. As with our previous tests, I will provide a one-page sheet with selected formulas. In particular this sheet will have formulas for the derivatives of the inverse trigonometric functions, and the double angle formulas for sines, cosines, and tangents. It will also have the statements of Simpson's Rule, the Divergence Test, the Alternating Series Test, and the Ratio Test.

Formulas you should know (and so they will not be on the sheet of formulas I provide):
- Derivative rules for exponential and power functions, the six trigonometric functions, the natural logarithm, the constant multiple rule, sums, differences, products, quotients, and the chain rule
- The Pythagorean Theorem, the definitions of the six trigonometric functions (in a right triangle, and with respect to the unit circle)
- Area formulas from geometry: area of squares, rectangles, triangles, trapezoids, and circles
- Volume of a rectangular box and a right circular cylinder (These are simple to remember as the area of the base times the height.)

For this test I'm planning to develop five to eight problems around the following themes:
1. Given a function, you should be able to set up a difference quotient for that function.
   a. You should be able to simplify the difference quotient to the point where you do not have a problem with having "zero in the denominator."
   b. You should be able to "take the limit" of the difference quotient, thus proving one of the derivative formulas.

2. Given a function, you should be able to calculate its derivative, and its antiderivative.
   a. Given the graph of a function, you should be able to sketch a graph of its derivative.
   b. Given a graph showing several functions, you should be able to identify graphs of the original function, its derivative, and its antiderivative.
   c. Given a graph of a function, you should be able to identify features of the graph of the derivative. In other words, looking at the graph of a function, you should be able to tell where its derivative will be zero, and identify intervals over which the derivative is positive or negative.
   d. Given a function, you should be able to find the family of antiderivatives. (Don't forget the "+ C" and what this means!)
   e. Take the derivative of a given function to verify that it is the antiderivative of another function.
   f. If two functions have the same derivative, what can you say about those functions? What does this have to do with the "+C"?

3. Given a function $f(x)$ represented as an expression or in a graph, you should be able to calculate the following kinds of limits:
   a. What happens as x approaches points where the function is undefined because it has a vertical asymptote or a hole?
   b. What happens at points where x approaches a jump discontinuity? (You have to consider the limit from each side – from the left and from the right, as these will not be the same.)
   c. What happens as x approaches ∞ or $-\infty$?

4. Given a region in the xy-plan that is bounded by a function, you should be able to find the area and the "net area" of the region
   a. using geometry, and
   b. using a definite integral.

5. Given a differential equation and an initial value:
   a. You should be able to find the family of antiderivatives from the differential equation.
   b. Then using the given initial value, you should be able to find the particular antiderivative that goes through the given point.

6. Given a function, you should be able to set up a Taylor polynomial of degree 6 to represent the function.

7. Given a series, you should be able to use the Alternating Series Test, the Ratio Test, and/or a comparison to a geometric series or harmonic series to determine whether the given series converges or diverges.

**Topics to be covered by this exam:**

Sketch graphs of functions:
You should be able to sketch or identify graphs of the following types of functions. Note that a, b, and c are constants. A sketch graph should clearly show x- and y-intercepts and the general shape of the graph.

$f(x) = ax + b$	$f(x) = b$	$f(x) = a\sqrt{x+b}$	$f(x) = ax^2 + bx$
$f(x) = ax^2 + b$	$f(x) = a/x$	$f(x) = a\sqrt{x} + b$	$f(x) = \sqrt{b - x^2}$
$f(x) = ax^3 + b$	$f(x) = a/x^2$	$f(x) = \sqrt{b + x^2}$	
$f(x) = a\sin(x) + b$	$f(x) = e^x$	$f(x) = a^x$, for $a > 0$	$f(x) = a^{-x}$, for $a > 0$
$f(x) = a\cos(x) + b$	$f(x) = \ln(x)$	$f(x) = (x+a)(x+b)(x+c)$	

Trigonometry
Algebra and trigonometry are prerequisites for this course. I assume that you know the definitions of the six basic trigonometric functions, and that you can use the two standard triangles and the unit circle to find exact values for angles related to the five angles of the standard triangles. In other words, you should be able to tell me that $\sin\left(\pi/3\right) = \sqrt{3}/2$ or that $\cos(\pi) = \cos(180°) = -1$.
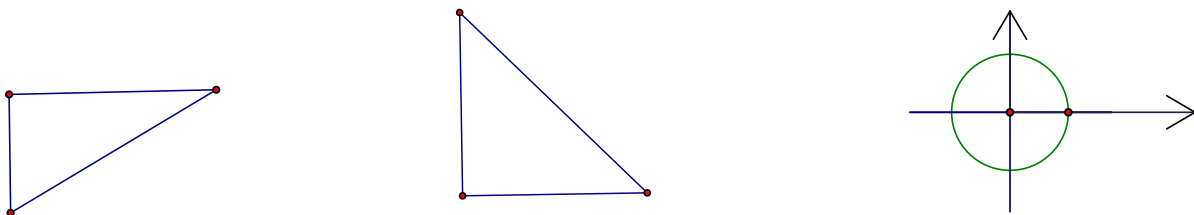
Difference quotients, definition of the derivative:
You should be able to set up a difference quotient, and simplify it to the point where there is no longer an "h" in the denominator. This is the first step in proving the various short-cut rules for computing derivatives. Now, you know how to take the limit as h $\rightarrow$ 0, and can complete the proof.

Calculating Derivatives:
You should be able to calculate the following kinds of derivatives:
- Constant function, linear function, power function (x^p, for an integer or rational number p)
- Polynomial function
- Each of the six trigonometric functions
- Exponential function: a^x, $a^{u(x)}$, e^x, $e^{u(x)}$. Note that if the exponent is anything more than simply "x", you need to apply the chain rule to complete the computation.
- Functions defined as a product or quotient
- Composite functions, which require the chain rule

Calculating Antiderivatives and Definite integrals
You should be able to calculate the following kinds of antiderivatives and definite integrals:
- Constant function, linear function
- Power function (x^p, for an integer or rational number p, for both $p \neq -1$, and for $p = -1$)
- Polynomial function
- The derivatives of each of the six trigonometric functions
- Exponential function (a^x, $a^{u(x)}$, e^x, $e^{u(x)}$)
- Functions requiring u-substitutions; i.e., functions whose derivative was found using the chain rule
- Functions requiring a partial fraction decomposition
- Functions requiring integration by parts ($u–dv$ substitution); that is, functions whose derivative was found using the product rule

Initial Value Problems
Given a differential equation, we can find the antiderivative, which gives us a whole family of functions. If we are given one point (the initial value or any other point), we can find the particular antiderivative which passes through that given point.

Early in the semester, we studied falling body problems. If an object such as a marble, a golf ball, or a baseball is thrown or dropped, gravity (g) is the acceleration force that causes an object to fall back to earth. The velocity of a falling object is the rate of change in its position, and its acceleration is the rate of change in its velocity. If you are given $a(t) = g$, you should be able to find expressions for the velocity function, $v(t)$, and the position function, $s(t)$. You should be able to use these three functions to answer questions like the following with regard to an object that is thrown or dropped:
- How high does it go?
- How long does it take to reach the ground?
- How long does it take to reach its maximum height?
- What is its maximum height?
- What is its velocity at its peak height?
- How fast is it moving at time t?
- … ? and so on.

The decision about whether down is a positive or negative direction is arbitrary, but it is important to be consistent throughout your solution to the problem. (See Note 1: Orientation in Section 5.2.)

Modeling with Differential Equations

A differential equation is an equation in which both a function and the derivative of that function appear. For example, throughout this course, we have seen that the differential equations $\frac{dy}{dx} = kx$ and $\frac{dy}{dx} = ky$ give rise to two entirely different families of functions. (See Activities 1, 2, and 3 of Section 3.1)   In Section 6.2, we learned the separation of variables technique, which makes it much clearer how these two different families of functions arise.

The solution to a differential equation is a family of functions (hence the constant C in these two families), the derivative of which gives you that differential equation. In other words, if *y(x)* is the solution to the differential equation *dy/dx* = ❏, then calculating *y'(x)* will give you ❏. Now that we know a variety of methods for finding antiderivatives, we can use these to help us solve differential equations.

Logistic Growth

Logistic growth is another situation involving a differential equation. In most settings, a population cannot grow larger and larger without bound; eventually the growth of the population becomes constrained by the natural environment. In this context, the growth rate (dy/dt) is proportional to both the current size of the population (y) and how far it is from its upper limit ($M - y$). That is, $\frac{dy}{dt} = k \cdot y \cdot (M - y)$. Solving this kind of differential equation requires using a partial fraction decomposition, and gives us an equation that involves the natural logarithm; hence constrained growth is modeled by a logistic equation.

Riemann sum, definition of the definite integral:

Given a function $f(x)$, you should be able to sketch a graph of the function, and then set up a left- or right-sum using rectangles with a small number of subdivisions ($n \leq 5$) and calculate this sum to get an estimate of the definite integral. You should be able to determine from your sketch whether this process gives an underestimate or overestimate of the definite integral.

The Fundamental Theorem of Calculus

If a function, f, is positive over an interval from a to b, the definite integral, $\int_a^b f(x)dx$, denotes the area of the region bounded by the graph of f and the x-axis between the vertical lines $x = a$ and $x = b$. Often, we can use geometry to find this area.

If the function is not positive over the entire interval but is sometimes positive and sometimes negative, the definite integral will give the *net area*. That is, where the function dips below the x-axis the "height" will be negative, so that the resulting "area" will be negative. If the part of the function that lies above the x-axis is symmetric with the part that lies below the x-axis, the net area will be "zero."

The Fundamental Theorem of Calculus tells us that we can use antiderivatives to calculate the value of the definite integral: $\int_a^b f(x)dx = F(b) - F(a)$, where $F(x)$ is the antiderivative of $f(x)$. Moreover, the derivative of the antiderivative will equal the original function, $\frac{d}{dx} \int f(x)dx, = f(x)$.

If $y = f(x)$ is a function plotted on an xy-coordinate graph, then the area bounded by the graph represents the product of x and y. This area then has meaning in the physical context of the problem. In Section 7.1.1, we worked with graphs of a temperature function; then by "adding up all the temperatures" and dividing by "the total interval of time" we calculated the average temperature. In

Section 7.1.2 – 7.1.3, we are worked with graphs of velocity (or speed) over time; so the area bounded by the graph represents the "sum of the velocity × time." In Section 7.1.4, the authors remind us it is not uncommon to use products to find <u>distance</u> (velocity × time), <u>area</u> (length × width), <u>volume</u> (height × area of base), <u>mass</u> (density × volume), and so on. In the days before we learned calculus, quantities like velocity (function of time), length (function of width), height (function of point on the base), and density (function of point in space) had to be *constant or linear functions* of their respective variables; then we could solve problems using simple arithmetic. In Chapter 7 we were able to consider situations in which these functions are not constant functions.

Of course, whenever possible, we will continue to use geometry to calculate areas. This is the core of the idea underlying the Area Function (in a handout), and the estimation of the area of Crater Lake. When the area that we seek is the area of a region bounded by some kind of irregular curve, so that geometry no longer does the job easily, we can divide up the region into smaller pieces. By making a finer and finer grid (Crater Lake) or smaller and smaller slices (Section 7.1.5), and adding up the areas of smaller pieces. Making the grid finer or the slices skinnier helps us to make the estimate of the area more accurate. These smaller pieces could be squares, rectangles, trapezoids, or triangles – all shapes for which we know how to calculate areas. In Sections 7.1.5 – 7.1.6, we get the formal definition of the definite integral as the sum of the areas of these smaller pieces.

Evaluating Limits
We need to calculate limits in situations in the following situations:
- o when evaluating a difference quotient to find the formula for the derivative of a function,
- o where there is some problem with the expression, such as a zero in the denominator,
- o when the function makes a "jump" at a particular value, or
- o when we are considering the ultimate value of an expression as the variable approaches positive or negative infinity.

- If there is a "zero in the denominator" you can often fix the problem algebraically. If you can eliminate the zero in the denominator, you have an expression that is identical to the original expression – except at the problem point.
  - o If you can fix the problem algebraically, this indicates that there is probably a "hole" in the graph at this point.
  - o If you cannot fix the problem algebraically, there is probably a vertical asymptote.
- A sketch graph of the function will often help us to "see" what is going on in the neighborhood of the problem points. You should be able to evaluate limits of functions given in a sketch graphs.
- In order to compute the limit of an expression as the variable goes to infinity, it often helps to think about what is happening as the variable gets larger (goes toward +∞) or gets smaller (goes toward -∞).
  - o A useful strategy is to consider the "largest terms," which are usually the terms with the highest powers of the variable as the other terms are "small" by comparison.
  - o To find the limit as x → ±∞ for a rational function, consider the highest powers of the variable in the numerator and the denominator. Then simplify the resulting expression.
- You should be able to calculate limits for constant functions, linear functions, polynomial functions, absolute value functions, and rational functions.
- You should be able to evaluate an improper integral. Calculating the value of an improper integral, $\int_a^\infty f(x)\, dx$, is a two-step process, which can be represented symbolically as $\lim_{T\to\infty} \int_a^T f(x)\, dx$. See Section 9.2.

Taylor Polynomials and Series
- You should be able to calculate the sum of a geometric series.
- You should be able to set up a Taylor polynomial for a given function. To do this you need to
  - take successive derivatives of the function,
  - evaluate each derivative at a given point (either at $x = 0$, or more generally at $x = a$),
  - then use these values as coefficients in a polynomial of the form
  $$y = a_0 + a_1 (x - a) + a_2 (x - a)^2 + a_3 (x - a)^3 + a_4 (x - a)^4 + \dots$$
- You should be able to identify whether a series is a geometric series, harmonic series, alternating series, or none of these.
  - If the series is a geometric series, you should be able to tell whether it converges or diverges. If it converges, you should be able to find the sum of a geometric series.
  - You should be able to use the Divergence Test to identify a series that does not converge.
  - You should be able to use the Alternating Series Test and Ratio Test to determine whether a given series converges or diverges.
  - **Comparison Test**: If you have a series whose terms (after some term a_k) are all smaller than a geometric series, you can conclude that the series converges. If you have a series whose terms (after some term a_k) are all larger than the corresponding terms of the harmonic series, you can conclude that the series diverges.

Good study questions: (You might be surprised how "easy" some of these exercises seem now!)
Chapter 10: *Sequences, Series, Taylor Polynomials, and Taylor Series*
- Exercises, Section 10.6: #1, 2, 4, 5, and 8. What do we learn by applying the Alternating Series Test or the Ratio Test? Can we compare these, term by term, to either a geometric series or the harmonic series?
- Exercises, Section 10.5: #1, 2, 3, 6, 7, and 8.
- Exercises, Section 10.3: #1 – 4
- Exercises, Section 10.2: #1, 5, 6, 7, 11, and 17. For Exercise #11, can you find the get the same sequence of functions by calculating the derivatives by hand? Can you evaluate these derivatives at a point, say x = 0, and put together the Taylor polynomial centered at x = 0?
- Exercises, Section 10.1: #1, 5, 6, and 7. How can you use the idea of geometric sums – see Example 1 in Section 10.1.1 – to simplify these calculations?

Chapter 9: *Evaluation of Improper Integrals, Limits as $x \to \infty$*
- Exrcises, Section 9.2: #11 – 12. Which of these definite integrals appear to converge as the upper limit → ∞?

Chapter 8: *Integral Calculus, Integration Rules*
- Exercises, Section 8.4: #1 – 4, 7, 8, 10 – 11
- Exercises, Section 8.3: #1 – 8, 10 – 11
- Exercises, Section 8.1: #1 – 2

Chapter 7: *Riemann Sums, The Fundamental Theorem of Calculus, Crater Lake*
- Exercises, Section 7.2: #1, 3, 4, 5, 8, 9, 15, and 20
- Exercises, Section 7.1: #1 – 10. You should be able to use both geometry and rules of anti-derivatives to do these exercises.

Chapter 6: *Antidifferentiation, Differential Equations, Separation of Variables, Sky Diving*
- <u>Exercises, Section 6.3</u>: #1 – 3 and 6 – 8
- <u>Exercises, Section 6.2</u>: #1 – 7
- <u>Exercises, Section 6.1</u>: #1 – 6
    - Remember finding "all antiderivatives" of each function involves a constant of integration; you can check your work by taking the derivative of your result, or by using Maple.
    - #3 – 4 (The strategy is to separate variables, take antiderivatives, and then use the given "initial value" to determine which one of the members of the family of antiderivatives is the desired particular solution.)
    - #6 More practice with antiderivatives, including trigonometric substitutions

- The problems on the Area Function handout
- Questions based on any of the projects – Sky Diving, Crater Lake, and Drug Dosage
- Questions from any of our earlier tests
- Web Work problem sets from the entire semester
- Study Questions for the Final Exam, which are provided on a separate handout